Problem A.1

Find the area enclosed be these three functions: $f(x) = 1$, $g(x) = x + 1$, $h(x) = 9 - x$

$g(x) = f(x) \Leftrightarrow x = 0; y = 1 \Rightarrow y = f(x)$ intersects $y = g(x)$ at $A(0;1)$

$f(x) = h(x) \Leftrightarrow x = 8; y = 1 \Rightarrow y = f(x)$ intersects $y = h(x)$ at $B(8;1)$

$g(x) = h(x) \Leftrightarrow x = 4; y = 5 \Rightarrow y = g(x)$ intersects $y = h(x)$ at $C(4;5)$

$$\left.\begin{array}{l} AC = \sqrt{(x_A - x_C)^2 + (y_A - y_C)^2} = \sqrt{4^2 + 4^2} = 4\sqrt{2} \\ BC = \sqrt{(x_B - x_C)^2 + (y_B - y_C)^2} = \sqrt{4^2 + 4^2} = 4\sqrt{2} \\ y = g(x) \text{ and } y = h(x) \text{ are two perpendicular lines} \end{array}\right\} \Rightarrow AC \perp BC$$

$$\Rightarrow S_{ABC} = \frac{AC \cdot BC}{2} = \frac{4\sqrt{2} \cdot 4\sqrt{2}}{2} = 16.$$

Problem A.2

Find the roots of the function $f(x) = 3^x \cdot (\log_2(x) - 3)^5 \cdot e^{x^2 - 3x}$.

$$\begin{cases} 3^x > 0 \quad \forall\, x \in \mathbb{R} & (1) \\ e^{x^2 - 3x} > 0 \quad \forall\, x \in \mathbb{R} & (2) \end{cases}$$

$$f(x) = 0 \Leftrightarrow 3^x \cdot (\log_2(x) - 3)^5 \cdot e^{x^2 - 3x} = 0 \quad (3)$$

$$\begin{aligned} (1); (2); (3) &\Rightarrow (\log_2(x) - 3)^5 = 0. \\ &\Rightarrow \log_2(x) - 3 = 0. \\ &\Rightarrow \log_2(x) = 3 \\ &\Rightarrow x = 8. \end{aligned}$$

Problem A.3

Find the derivative $f'(x)$ of the function $f(x) = x^{\sin(x)}$.

$$(\ln f(x))' = \frac{f'(x)}{f(x)}$$

$$\ln f(x) = \ln x^{\sin x} = \sin x . \ln x$$

$$\rightarrow (\ln f(x))' = (\sin x . \ln x)' = (\sin x)' . \ln x + \sin x . (\ln x)'$$

$$= \cos x . \ln x + \sin x . \frac{1}{x} .$$

$$\circledast \ (\ln f(x))' = \frac{f'(x)}{f(x)}$$

$$\Rightarrow f'(x) = f(x) . (\ln (f(x)))'$$

$$= x^{\sin x} . \left(\cos x . \ln x + \frac{\sin x}{x} \right)$$

Problem A.4

Find the value of this expression for $n \to \infty$:

$$\left(\sqrt{1 - \frac{1}{n}}\right)^n \cdot \sqrt{\left(1 - \frac{1}{n}\right)^n}$$

Hint: You may use that $e^x = (1 + \frac{x}{n})^n$ for $n \to \infty$.

$$\left(\sqrt{1 - \frac{1}{n}}\right)^n . \sqrt{\left(1 - \frac{1}{n}\right)^n}$$

$$= \left[\left(1 - \frac{1}{n}\right)^{\frac{1}{2}}\right]^n . \left[\left(1 - \frac{1}{n}\right)^n\right]^{\frac{1}{2}}$$

$$= \left(1 - \frac{1}{n}\right)^{\frac{n}{2}} . \left(1 - \frac{1}{n}\right)^{\frac{n}{2}}$$

$$= \left(1 - \frac{1}{n}\right)^n = e^{-1} \quad (\text{in this case } x = -1)$$

Problem B.1

Find all positive integers n such that $n^4 - 1$ is divisible by 5.

$n^4 - 1 \vdots 5.$

$\Rightarrow (n^2+1)(n-1)(n+1) \vdots 5.$

$\Rightarrow \left[\begin{array}{l} n^2+1 \vdots 5 \\ n-1 \vdots 5 \\ n+1 \vdots 5 \end{array}\right.$

$\Rightarrow \left[\begin{array}{l} n \equiv 2 \pmod 5 \\ n \equiv 1 \pmod 5 \\ n \equiv 4 \pmod 5 \end{array}\right.$

Problem B.2

Prove the following inequality between the harmonic, geometric, and arithmetic mean with $x, y \geq 0$:

$$\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \leq \frac{x+y}{2}$$

(*) $\frac{x+y}{2} \geq \sqrt{xy} \Leftrightarrow x+y \geq 2\sqrt{xy}. \Leftrightarrow (\sqrt{x})^2 - 2\sqrt{xy} + (\sqrt{y})^2 \geq 0.$

$\Leftrightarrow (\sqrt{x}-\sqrt{y})^2 \geq 0$ (true for all $x, y \geq 0$)

$\Rightarrow \frac{x+y}{2} \geq \sqrt{xy}$ is true with $x, y \geq 0$ (1)

(*) $\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \Leftrightarrow \sqrt{xy} \cdot \frac{x+y}{xy} \geq 2. \Leftrightarrow x+y \geq 2\sqrt{xy}.$

$\Leftrightarrow (\sqrt{x}-\sqrt{y})^2 \geq 0$ (true for all $x, y \geq$

$\Rightarrow \frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy}$ is true with $x, y \geq 0$ (2)

(1); (2) $\Rightarrow \frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \leq \frac{x+y}{2}$ with $x, y \geq 0$

Problem B.3

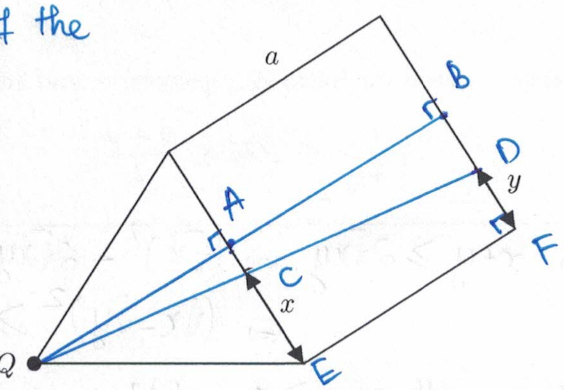
Suppose you have to distribute the numbers $\{1, 2, 3, \ldots, 2n-1, 2n\}$ over n buckets. Show that there will always be at least one bucket with its sum of numbers to be $\geq 2n+1$.

According to the distribution, there will be n buckets with 2 numbers in each bucket.
If all buckets have its sum of number to be $< 2n+1$
$\Rightarrow$ There exists a bucket with its sum of numbers: $2n + x < 2n+1$ (1)
(x is a number in the given list)
However, ~~in~~ the given list is an ascending list of numbers and the least number is 1 $\Rightarrow x \geq 1 \Rightarrow 2n + x \geq 2n+1$ (which contradicts (1))
$\Rightarrow$ There will always be at least one bucket with its sum of numbers to be $\geq 2n+1$

Problem B.4

Consider an equal-sided triangle connected to a square with side a (see drawing). A straight line from Q intersects the square at x and y. You have given x, find an equation for the intersection at $y(x)$.

A, B are mid points of the square side.
$CE = x$; $DF = y$.



Thales theorem in ΔQBD:

$AC \parallel BD \Rightarrow \frac{AC}{BD} = \frac{QA}{QB}$.

$\frac{AC}{BD} = \frac{\frac{a}{2}-x}{\frac{a}{2}-y}$; $\frac{QA}{QB} = \frac{\frac{a\sqrt{3}}{2}}{\frac{a\sqrt{3}}{2}+a}$

$$\Rightarrow \frac{\frac{a}{2}-x}{\frac{a}{2}-y} = \frac{\frac{a\sqrt{3}}{2}}{\frac{a\sqrt{3}}{2}+a}$$

$$\Rightarrow \frac{\frac{a}{2}-x}{\frac{a}{2}-y} = \frac{\sqrt{3}}{\sqrt{3}+2}$$

$$\Rightarrow \frac{a}{2} - y = \frac{(\frac{a}{2}-x)(\sqrt{3}+2)}{\sqrt{3}}$$

$$\Rightarrow y = \frac{a}{2} - \frac{(\frac{a}{2}-x)(\sqrt{3}+2)}{\sqrt{3}}$$

$$\Rightarrow y = \frac{a\sqrt{3} - 2(\frac{a}{2}-x)(\sqrt{3}+2)}{2\sqrt{3}}$$

$$\Rightarrow y = \frac{a\sqrt{3} - 2(\frac{a\sqrt{3}}{2} + a - \sqrt{3}x - 2x)}{2\sqrt{3}}$$

$$\Rightarrow y = \frac{-2a + 2\sqrt{3}x + 4x}{2\sqrt{3}}$$

$$\Rightarrow \boxed{y = \frac{(\sqrt{3}+2)x - a}{\sqrt{3}}}$$

Problem C.1

The sum of divisor function $\sigma(n)$ returns the sum of all divisors d of the number n:

$$\sigma(n) = \sum_{d|n} d$$

We denote N_k any number that fulfils the following condition:

$$\sigma(N_k) \geq k \cdot N_k$$

Find examples for N_3, N_4, N_5 and prove that they fulfil this condition.

Examples: $N_3 = 5!$
$N_4 = 11!$

⊛ $N_3 = 5! = 2^3 . 3 . 5$

→ The sum of divisors of $5! = (2^3 + 2^2 + 2^1 + 2^0) \times (3 + 3^0) \times (5 + 5^0) = 36$

$360 = 3 . 5!$

→ 5! fulfils this condition

⊛ $N_4 = 11! = 2^8 . 3^4 . 5^2 . 7 . 11$

→ The sum of divisors of $11! = (2^8 + 2^7 + \dots + 2^1 + 2^0) \times (3^4 + 3^3 + \dots + 3^0)$
$\times (5^2 + 5^1 + 5^0) \times (7 + 7^0) \times (11 + 11^0)$
$= 184009056$

We have: $184009056 > 4 . 11!$

→ 11! fulfils this condition

Problem C.2

This problem requires you to read following recently published scientific article:

Encoding and Visualization in the Collatz Conjecture.
George M. Georgiou, arXiv:1811.00384, (2019).
Link: `https://arxiv.org/pdf/1811.00384.pdf`

Please answer following questions related to the article:

(a) Explain the *Collatz conjecture* in your own words. Have we proven this conjecture?

Collatz conjecture is a function with the initial input of any positive integer and the output will aways be 1 after some applications of the function. Eventually the cycle $1 \to 4 \to 2 \to 1$ will be repeated. We have not proven this conjecture

(b) What is the $C(n)$ cycle and the $T(n)$ cycle of the number $n = 48$?

$C(48)$ cycle: $1 \to 4 \to 2 \to 1$

$T(48)$ cycle: $1 \to 2 \to 1$

(c) Explain the meaning of $\sigma_\infty(n)$ and calculate $\sigma_\infty(104)$.

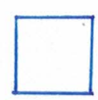
$\sigma_\infty(n)$: the least k application of T that makes the sequence of iteration reach 1 for the first time.

Apply $T^k(104)$ we have: $104 \to 52 \to 26 \to 13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1 \to \dots$
$\to \sigma_\infty(104) = 10$

(d) Find the binary encoding of $n = 32, 53, 80$ and explain why they all start with "111".

Binary encoding of $n = 32$ is 11111, of $n = 53$ is 11011110, of $n=80$ is 1110111
Because $32; 53; 80 > 5 \to$ Their iteration will end in $\underset{(1)}{8} \to \underset{(1)}{4} \to \underset{(1)}{2} \to 1$ } they start with "111"
The bitsare produced from right to left

(e) Make a drawing of the Collatz curve of $n = 2^{10} = 1024$.

The curves for $n = 2^k$ $(k \geq 4)$ will be a square. In this case $k = 10 > 4$
$\to$ The curve of the Collatz curve of $n = 2^{10}$ is a square: 

(f) What is more common according to the data: r-curves with finite girth or acyclic r-curves?

r-curves with finite girth is more common