Problem A.1

Find all points (x, y) where the functions $f(x), g(x), h(x)$ have the same value:

$$f(x) = 2^{x-5} + 3, \quad g(x) = 2x - 5, \quad h(x) = \frac{8}{x} + 10$$

Step 1: Find all points (x, y) where the functions $g(x), h(x)$ have the same value

Consider the equation: $g(x) = h(x) \Leftrightarrow 2x - 5 = \frac{8}{x} + 10$

$$\Leftrightarrow \begin{cases} 2x^2 - 5x = 8 + 10x \\ x \neq 0 \end{cases} \Leftrightarrow \begin{cases} 2x^2 - 15x - 8 = 0 \\ x \neq 0 \end{cases} \Leftrightarrow \begin{cases} (x-8)(2x+1) = 0 \\ x \neq 0 \end{cases} \Leftrightarrow \left[\begin{array}{l} x = 8 \\ x = -\frac{1}{2} \end{array} \right.$$

At $x = 8$: $g(8) = h(8) = 11$. At $x = -\frac{1}{2}$: $g(-\frac{1}{2}) = h(-\frac{1}{2}) = -6$

Thus, $(8, 11)$ and $(-\frac{1}{2}, -6)$ are all points where $g(x), h(x)$ have the same value

Step 2: We will check if $f(x)$ also has the same value as $g(x)$ and $h(x)$ at $x = 8$ and $x = -\frac{1}{2}$: We have: $f(8) = 2^{8-5} + 3 = 11$, $f(-\frac{1}{2}) = 2^{-\frac{1}{2}-5} + 3 > 0 + 3 = 3 > -6$

Step 3: Therefore, $(8, 11)$ is the point where $f(x), g(x), h(x)$ have the same value.

Problem A.2

Determine the roots of the function $f(x) = (5^{2x} - 6)^2 - (5^{2x} - 6) - 12$.

Step 1: We consider the equation: $f(x) = 0 \Leftrightarrow (5^{2x} - 6)^2 - (5^{2x} - 6) - 12 = 0 \; (1)$

Step 2: We use change of variables: Let $t = 5^{2x} - 6$. The equation (1) becomes:

$$t^2 - t - 12 = 0 \quad (2)$$

Step 3: Solve (2) as an equation for new variable t.

$$(2) \Leftrightarrow (t^2 - 4t) + (3t - 12) = 0 \Leftrightarrow t(t-4) + 3(t-4) = 0 \Leftrightarrow (t+3)(t-4) = 0$$

$$\Leftrightarrow \left[\begin{array}{l} t + 3 = 0 \\ t - 4 = 0 \end{array} \right. \Leftrightarrow \left[\begin{array}{l} t = -3 \\ t = 4 \end{array} \right. \quad (*)$$

Step 4: We will solve (1) as we replace $t = 5^{2x} - 6$ at $(*)$

$$\left[\begin{array}{l} t = -3 \\ t = 4 \end{array} \right. \Leftrightarrow \left[\begin{array}{l} 5^{2x} - 6 = -3 \\ 5^{2x} - 6 = 4 \end{array} \right. \Leftrightarrow \left[\begin{array}{l} 5^{2x} = 3 \\ 5^{2x} = 10 \end{array} \right. \Leftrightarrow \left[\begin{array}{l} 2x = \log_5 3 \\ 2x = \log_5 10 \end{array} \right. \Leftrightarrow \left[\begin{array}{l} x = \frac{1}{2} \cdot \log_5 3 \\ x = \frac{1}{2} \cdot \log_5 10 \end{array} \right.$$

Step 5: Since all the equations and systems above are equivalent, we conclude that the roots of the function $f(x)$ are $\frac{1}{2} \log_5 3$ and $\frac{1}{2} \log_5 10$

Problem A.3

Find the derivative $f'_m(x)$ of the following function with respect to x:

$$f_m(x) = \left(\sum_{n=1}^{m} n^x \cdot x^n\right)^2$$

Step 1: Let $g_m(x) = \sum_{n=1}^{m} n^x . x^n$, then $f_m(x) = (g_m(x))^2$

Thus $f_m(x)$ is a composite function: $f'_m(x) = 2 g_m(x) . g'_m(x)$

Step 2: The Derivatie of a sum of functions is equal to the sum of the derivatives of those functions. Hence: $g'_m(x) = \sum_{n=1}^{m} (n^x . x^n)'$

Step 3: For $n = 1, 2, \ldots, m$, $n^x . x^n$ is the product of two functions of x.

Thus: $(n^x . x^n)' = (n^x)' . x^n + n^x . (x^n)' = n^x . \ln n . x^n + n^x . n . x^{n-1} = \ln n . n^x . x^n + n^{x+1} . x^{n-1}$

Step 4: Therefore: $g'_m(x) = \sum_{n=1}^{m} (\ln n . n^x . x^n + n^{x+1} . x^{n-1})$ and:

$$f'_m(x) = 2 . \left(\sum_{n=1}^{m} n^x . x^n\right)\left(\sum_{n=1}^{m} (\ln n . n^x . x^n + n^{x+1} . x^{n-1})\right)$$

Problem A.4

Find at least one solution to the following equation:

$$\frac{\sin(x^2-1)}{1-\sin(x^2-1)} = \sin(x) + \sin^2(x) + \sin^3(x) + \sin^4(x) + \cdots$$

Step 1: We have RHS $= \sin(x) . \left(1 + \sin(x) + \sin^2(x) + \sin^3(x) + \ldots\right)$

Then, we realize that $1 + \sin(x) + \sin^2(x) + \sin^3(x) + \ldots$ is the infinite sum of a geometric series, with the common ratio as $\sin(x)$

So, for $\sin(x) \neq 1$, or $x \neq \frac{\pi}{2} + k.2\pi$, then: RHS $= \sin(x) . \frac{1}{1-\sin(x)} = \frac{\sin(x)}{1-\sin(x)}$

Step 2: The equation becomes: $\frac{\sin(x^2-1)}{1-\sin(x^2-1)} = \frac{\sin(x)}{1-\sin(x)}$

We see that if $x^2 - 1 = x \Rightarrow \sin(x^2-1) = \sin(x) \Rightarrow \frac{\sin(x^2-1)}{1-\sin(x^2-1)} = \frac{\sin(x)}{1-\sin x}$, if $\sin(x) \neq 0$

Step 3: Therefore, the solutions of the equation: $x^2 - 1 = x$ (1) also satisfies the original equation, if $\sin(x) = 0$. Since (1) has a solution $x = \frac{1+\sqrt{5}}{2}$ and $\sin\left(\frac{1+\sqrt{5}}{2}\right) \neq 0$, the original equation has at least a solution $x = \frac{1+\sqrt{5}}{2}$

Problem B.1

Consider the following sequence of successive numbers of the 2^k-th power:

$$1,\ 2^{2^k},\ 3^{2^k},\ 4^{2^k},\ 5^{2^k},\ \ldots$$

Show that the difference between the numbers in this sequence is odd for all $k \in \mathbb{N}$.

Step 1: Let $a(n)$ be the number at the n-th place in this sequence

Then $a(n) = n^{2^k}$, for all $n \in \mathbb{N}$

Step 2: For all $k \in \mathbb{N}$: 2^k is a positive integer number and $2^k \geq 2$.

If n is odd: $a(n) = \underbrace{n.n.n.....n}_{2^k \text{ times}}$ is still an odd number. Similarly, if n is even, then $a(n)$ is still an even number. (I forgot to mention $n \in \mathbb{N}$, for $a(n)$ to be defined)

Step 3: Let n be an arbitrary positive integer. Then, n can only be odd or even.

If n is odd, then $n+1$ is even. Thus, $a(n)$ is odd and $a(n+1)$ is even $\Rightarrow a(n) - a(n+1)$ is odd

If n is even, then $n+1$ is odd. Thus, $a(n)$ is even and $a(n+1)$ is odd $\Rightarrow a(n) - a(n+1)$ is odd

Therefore: Q.E.D.

Problem B.2

Prove this identity between two infinite sums (with $x \in \mathbb{R}$ and $n!$ stands for factorial):

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

Step 1: We have the ~~Mac~~ Maclaurin expansion formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$$

for any function $f(x)$ that is infinitely differentiable at 0

Step 2: Choose $f(x) = e^x$. Then, as we know $f(x) = f'(x) = f^{(2)}(x) = \ldots = f^{(n)}(x) = e^x$

Replace e^x into the Maclaurin expansion formula we obtain:

$$e^x = \sum_{n=0}^{\infty} \frac{e^0}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (1)$$

Step 3: Hence $\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 = (e^x)^2 = e^{2x}$

Replace x as $2x$ in (1) we have: $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$

Therefore, we have the desired identity, since both sides are equal to e^{2x}

Problem B.3

You have given a function $\lambda : \mathbb{R} \to \mathbb{R}$ with the following properties ($x \in \mathbb{R},\ n \in \mathbb{N}$):

$$\lambda(n) = 0\ ,\quad \lambda(x+1) = \lambda(x)\ ,\quad \lambda\left(n + \frac{1}{2}\right) = 1$$

Find two functions $p, q : \mathbb{R} \to \mathbb{R}$ with $q(x) \neq 0$ for all x such that $\lambda(x) = q(x)(p(x)+1)$.

Step 1 : We will first find some properties of $\lambda(x)$

First we have : $\lambda(1) = 0$, $\lambda\left(\frac{3}{2}\right) = \lambda\left(1+\frac{1}{2}\right) = 1$, since $\lambda(n) = 0$, $\lambda\left(n+\frac{1}{2}\right) = 1$, for all $n \in \mathbb{N}^*$

Because $\lambda(x+1) = \lambda(x)$, for all $x \in \mathbb{R}$, we can obtain:

$$\lambda(0) = \lambda(1) = 0\ ,\quad \lambda\left(\tfrac{1}{2}\right) = \lambda\left(\tfrac{3}{2}\right) = 1$$

Applying $\lambda(x) = \lambda(x+1)$ n times ($n \in \mathbb{N}$) we have: $\lambda(x) = \lambda(x+1) = \lambda(x+2) = \dots = \lambda(x+n)$ for $x \in \mathbb{R}$. Replace x as $x-n$, then $\lambda(x-n) = \lambda(x)$, for all $x \in \mathbb{R}$, $n \in \mathbb{N}$

Combining the two properties , we have : $\lambda(x) = \lambda(x+n)$, for all $x \in \mathbb{R}$, n is an integer

Then for any $x \in \mathbb{R}$, we have $x = a + b$, for a is the greatest integer not greater than x, and $b \in [0,1)$. Following the above property: $f(x) = f(a+b) = f(b)$

That means we can determine the value $f(x)$ at every $x \in \mathbb{R}$ when we can determine the value $f(x)$ at every $x \in [0,1)$.

Step 2: We will next establish some properties of $p(x)$ and $q(x)$

Since $\lambda(x) = q(x).(p(x)+1)$, let $x = 0$ then: $q(0).(p(0)+1) = \lambda(0) = 0$

Because $q(0) \neq 0 \Rightarrow p(0) + 1 = 0 \Rightarrow p(0) = -1$

$q(0)$ can be any value different from zero. For simplicity , we let $q(0) = 1$

Similarly , if there exists x such that $\lambda(x) = 0$, we have $p(x) = -1$ and can choose $q(x) = 1$.

Now; ~~q(0)~~ $q\left(\frac{1}{2}\right).\left(p\left(\frac{1}{2}\right)+1\right) = \lambda\left(\frac{1}{2}\right) = 1$. We can choose $p\left(\frac{1}{2}\right) = 0$, then we obtain $q\left(\frac{1}{2}\right) = 1$

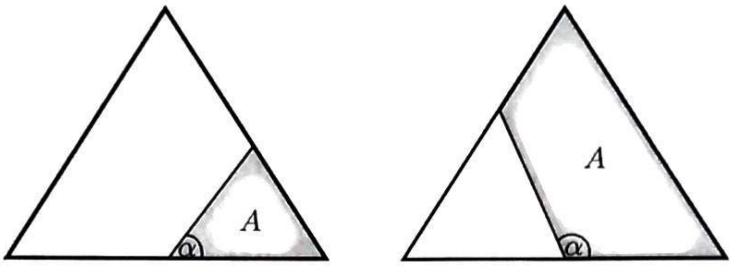
Similarly, if $\lambda(x) \neq 0$, we can choose $p(x) = 0$, then $q(x) = \lambda(x) \neq 0$.

Step 3: Therefore , we can choose two functions p, q that satisfy above conditions

$$p(x) = \begin{cases} -1\ , \text{ if } \lambda(x) = 0 \\ 0\ , \text{ if } \lambda(x) \neq 0 \end{cases} \quad \text{and} \quad q(x) = \begin{cases} 1, \text{ if } \lambda(x) = 0 \\ \lambda(x), \text{ if } \lambda(x) \neq 0 \end{cases}$$

Problem B.4

You have given an equal sided triangle with side length a. A straight line connects the center of the bottom side to the border of the triangle with an angle of α. Derive an expression for the enclosed area $A(\alpha)$ with respect to the angle (see drawing).



<u>Step 1</u>: Name the triangle ABC, M is the center of BC. We have 3 cases to consider

<u>Step 2</u>: Case 1: $0<\alpha<\frac{\pi}{2}$ and the straight line through M cuts AC at N.

$$A(\alpha) = A(MCN)$$

We draw the height NH of the triangle MCN

Consider the triangle CHN that has a right angle $\widehat{CHN}$.

Then $\frac{NH}{CH} = \tan\widehat{NCH} = \tan\frac{\pi}{3} = \sqrt{3} \Rightarrow CH = NH.\frac{1}{\sqrt{3}}$ ($\widehat{NCH} = \frac{\pi}{3}$ since ABC is an equal sided triangle)

Similarly: $MH = NH.\frac{1}{\tan\widehat{NMH}} = \frac{NH}{\tan\alpha}$

Since $CH + MH = CM = \frac{a}{2} \Rightarrow NH\left(\frac{1}{\sqrt{3}}+\frac{1}{\tan\alpha}\right) = \frac{a}{2} \Rightarrow NH = \frac{a}{2\left(\frac{1}{\sqrt{3}}+\frac{1}{\tan\alpha}\right)}$

Therefore: $A(MCN) = \frac{1}{2}.NH.MC = \frac{1}{2}\cdot\frac{a}{2\left(\frac{1}{\sqrt{3}}+\frac{1}{\tan\alpha}\right)}\cdot\frac{a}{2} = \frac{a^2}{8\left(\frac{1}{\sqrt{3}}+\frac{1}{\tan\alpha}\right)}$

<u>Step 3</u>. Case 2: $\alpha = \frac{\pi}{2}$ and the straight line through M is through A

Then AM is not only a median but also the height.

Similar as above, we have $\frac{AM}{CM} = \sqrt{3} \Rightarrow AM = CM.\sqrt{3} = \frac{a\sqrt{3}}{2}$

Then $A(ABC) = \frac{1}{2}BC.AM = \frac{1}{2}.a.\frac{a\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{4}$; $A(\alpha) = A(AMC) = \frac{1}{2}.MC.AM = \frac{1}{2}\cdot\frac{a}{2}\cdot\frac{a\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{8}$

<u>Step 4</u>: Case 3: $\frac{\pi}{2}<\alpha<\pi$ and the straight line through M cuts AB at N

Then $A(\alpha) = A(ABC) - A(MBN) = \frac{a^2\sqrt{3}}{4} - \frac{a^2}{8\left(\frac{1}{\sqrt{3}}+\frac{1}{\tan(\pi-\alpha)}\right)} = \frac{a^2\sqrt{3}}{4} - \frac{a^2}{8\left(\frac{1}{\sqrt{3}}-\frac{1}{\tan\alpha}\right)}$

<u>The way we calculate A(MBN) is similar to step 1, now with angle $\pi-\alpha$ instead of α</u>

Problem C.1

Let $\pi(N)$ be the number of primes less than or equal to N (example: $\pi(100) = 25$). The famous prime number theorem then states (with $\sim$ meaning *asymptotically equal*):

$$\pi(N) \sim \frac{N}{\log(N)}$$

Proving this theorem is very hard. However, we can derive a statistical form of the prime number theorem. For this, we consider *random primes* which are generated as follows:

(i) Create a list of consecutive integers from 2 to N.

(ii) Start with 2 and mark every number > 2 with a probability of $\frac{1}{2}$.

(iii) Let n be the next non-marked number. Mark every number $> n$ with a probability of $\frac{1}{n}$.

(iv) Repeat (iii) until you have reached N.

All the non-marked numbers in the list are called *random primes.*

(a) Let q_n be the probability of n being selected as a *random prime* during this algorithm. Find an expression for q_n in terms of q_{n-1}.

(b) Prove the following inequality of q_n and q_{n+1}:

$$\frac{1}{q_n} + \frac{1}{n} < \frac{1}{q_{n+1}} < \frac{1}{q_n} + \frac{1}{n-1}$$

(c) Use the result from (b) to show this inequality:

$$\sum_{k=1}^{N} \frac{1}{k} < \frac{1}{q_N} < \sum_{k=1}^{N} \frac{1}{k} + 1$$

(d) With this result, derive an asymptotic expression for q_n in terms of n.

(e) Let $\tilde{\pi}(N)$ be the number of *random primes* less than or equal to N. Use the result from (d) to derive an asymptotic expression for $\tilde{\pi}(N)$, i.e. the prime number theorem for *random primes.*

(a) We have 2 cases to consider:

Case 1: $n-1$ is a random prime, i.e it is not marked. This case happens with the probability of q_{n-1}: Then, n goes through the same steps at (iii) as $n-1$, after that n goes through one more step (iii) as $n-1$ is the next non-marked number

When n goes through the same steps as $n-1$, the probability for n to not be marked is ~~with a pro~~ the same as $n-1$ and equals q_{n-1}

For the last step when we consider $n-1$ to be the next non-marked number, n is

marked with a probability of $\frac{1}{n-1}$. Hence, n is not marked with a probability of $1-\frac{1}{n-1}$.
Therefore, the probability for n to not be marked in this case is $q_{n-1}\left(1-\frac{1}{n-1}\right)$, using the probability multiplication formula.

Case 2: $n-1$ is not a random prime. This case happens with the probability of $1-q_{n-1}$. In this case, the number n goes through the same steps at (iii) as $n-1$. Thus, n is not marked with the probability of q_{n-1}

Combining these two cases, we obtain the probability for n to be a random prime: $q_n = q_{n-1}\cdot q_{n-1}\left(1-\frac{1}{n-1}\right)+(1-q_{n-1})q_{n-1} = q_{n-1}\left(1-\frac{q_{n-1}}{n-1}\right)$, for all $n\in\mathbb{N}, n\geq 3$

(b) Step 1: From the formula, we have: $q_{n+1} = q_n\left(1-\frac{q_n}{n}\right)$, for all $n\in\mathbb{N}, n\geq 2$

Then we have: $\frac{1}{q_{n+1}}-\frac{1}{q_n} = \frac{q_n-q_{n+1}}{q_{n+1}\cdot q_n} = \frac{q_n-q_n\left(1-\frac{q_n}{n}\right)}{q_{n+1}\cdot q_n} = \frac{\frac{q_n^2}{n}}{q_{n+1}\cdot q_n} = \frac{q_n}{n\cdot q_{n+1}}$

Step 2: We will prove the left inequality. $\frac{1}{q_n}+\frac{1}{n}<\frac{1}{q_{n+1}} \Leftrightarrow \frac{1}{q_{n+1}}-\frac{1}{q_n}>\frac{1}{n}$

$$\Leftrightarrow \frac{q_n}{n\cdot q_{n+1}}>\frac{1}{n} \Leftrightarrow q_n > q_{n+1} \Leftrightarrow q_n > q_n\left(1-\frac{q_n}{n}\right) \Leftrightarrow \frac{q_n^2}{n}>0$$

The last inequality is true, since $q_n>0$, for all $n\in\mathbb{N}, n\geq 2$

Step 3: We will prove the right inequality: $\frac{1}{q_n}+\frac{1}{n-1}>\frac{1}{q_{n+1}} \Leftrightarrow \frac{1}{q_{n+1}}-\frac{1}{q_n}<\frac{1}{n-1}$

$$\Leftrightarrow \frac{q_n}{n\cdot q_{n+1}}<\frac{1}{n-1} \Leftrightarrow q_n < \frac{n}{n-1}\cdot q_{n+1} \Leftrightarrow q_n < \frac{n}{n-1}\cdot q_n\left(1-\frac{q_n}{n}\right)$$

$$\Leftrightarrow 1<\frac{n}{n-1}\left(1-\frac{q_n}{n}\right) \text{ (since } q_n>0) \Leftrightarrow 1<\frac{n}{n-1}-\frac{q_n}{n-1} \Leftrightarrow \frac{q_n}{n-1}<\frac{1}{n-1} \Leftrightarrow q_n<1$$

The last inequality is true, since $q_n<1$, for all $n\in\mathbb{N}, n\geq 3$.

There is a special case where $n=2$, then $\frac{1}{q_3}=\frac{1}{\frac{1}{2}}=2$, $\frac{1}{q_2}+\frac{1}{2-1}=\frac{1}{1}+\frac{1}{1}=2$.

Step 4: Therefore, we can conclude:

$$\frac{1}{q_n}+\frac{1}{n}<\frac{1}{q_{n+1}}<\frac{1}{q_n}+\frac{1}{n-1}, \text{ for all } n\in\mathbb{N}, n\geq 3.$$

Problem C.1

(c) Step 1 : From the inequality we obtain in (B), we have:

$$\frac{1}{q_2} + \frac{1}{2} < \frac{1}{q_3}$$

$$\frac{1}{q_3} + \frac{1}{3} < \frac{1}{q_4}$$

$$\cdots$$

$$\frac{1}{q_{N-1}} + \frac{1}{N-1} < \frac{1}{q_N}$$

Adding all these inequalities and eliminating the numbers that appear on both sides we obtain: $\frac{1}{q_N} > \frac{1}{q_2} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N-1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N-1} = \sum_{k=1}^{N-1} \frac{1}{k}$

Since $q_2 = 1$

Step 2: Also from (B), we have:

$$\frac{1}{q_3} = \frac{1}{q_2} + 1$$

$$\frac{1}{q_4} < \frac{1}{q_3} + \frac{1}{2}$$

$$\cdots$$

$$\frac{1}{q_N} < \frac{1}{q_{N-1}} + \frac{1}{N-2}$$

Adding all these ~~numbers~~ and eliminating the numbers that appear on both sides we obtain: $\frac{1}{q_N} < \frac{1}{q_2} + 1 + \frac{1}{2} + \dots + \frac{1}{N-2} < \sum_{k=1}^{N} \frac{1}{k} + 1$ (Since $q_2 = 1$)

Step 3 : Combining two above results, we get a pretty close result to the desired inequality:

$$\sum_{k=1}^{N-1} \frac{1}{k} < \frac{1}{q_N} < \sum_{k=1}^{N} \frac{1}{k} + 1$$

Problem C.1

(d) Step 1: We apply the famous result : $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \sim \log(n)$ and the fact that $\lim_{n \to +\infty} \log(n) = +\infty$, we obtain:

$$\sum_{k=1}^{N} \frac{1}{k} \sim \ln(N) \quad \text{and} \quad \sum_{k=1}^{N} \frac{1}{k} + 1 \sim \log(N)$$

Step 2 : The result from (c) states that

$$\sum_{k=1}^{N} \frac{1}{k} < \frac{1}{q_N} < \sum_{k=1}^{N} \frac{1}{k} + 1$$

Therefore : $\frac{1}{q_N} \sim \log(N) \iff q_N \sim \frac{1}{\log(N)}$

Step 3 : Hence, we have derived an asymptotic expression for q_n in terms of n, which is $\frac{1}{\log(n)}$

(e) Step 1: From the definition of q_n and $\tilde{\pi}(N)$, we can obtain the formula:

$$\tilde{\pi}(N) = q_2 + q_3 + \dots + q_N$$

Step 2: Using the result from (d), we get

$$\tilde{\pi}(N) \sim \sum_{k=2}^{N} \frac{1}{\log(k)}$$

Problem C.2

This problem requires you to read following scientific article:

On the harmonic and hyperharmonic Fibonacci numbers.
Tuglu, N., Kızılateş, C. & Kesim, S. Adv Differ Equ (2015).
Link: https://doi.org/10.1186/s13662-015-0635-z

Use the content of the article to work on the problems (a-f) below. All problems marked with * are bonus problems (g-i) that can give you extra points. However, it is not possible to get more than 40 points in total.

(a) What are the values of H_n, F_n and $\mathbb{F}_n$ for $n = 1, 2, 3$?

Step 1 : $H_1 = 1$, $H_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

Step 2 : $F_1 = 1$, $F_2 = F_0 + F_1 = 1$, $F_3 = F_1 + F_2 = 2$

Step 3 : $\mathbb{F}_1 = 1$, $\mathbb{F}_2 = 1 + 1 = 2$, $\mathbb{F}_3 = 1 + 1 + \frac{1}{2} = \frac{5}{2}$

(b) Determine the hyperharmonic number $H_3^{(10)}$ (Tip: use Equation 4) and $F_2^{(3)}$.

Step 1 : $H_3^{(10)} = \sum_{t=1}^{3} \binom{3+10-t-1}{10-1} \cdot \frac{1}{t} = \binom{11}{9} \cdot 1 + \binom{10}{9} \cdot \frac{1}{2} + \binom{9}{9} \cdot \frac{1}{3} = 55 + 10 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} = \frac{181}{3}$

Step 2 : We have $F_2^{(1)} = F_0^{(0)} + F_1^{(0)} + F_2^{(0)} = 0 + 1 + 1 = 2$, $F_2^{(2)} = F_0^{(1)} + F_1^{(1)} + F_2^{(1)} = 0 + 1 + 2 = 3$

Hence $F_2^{(3)} = F_0^{(2)} + F_1^{(2)} + F_2^{(2)} = 0 + 1 + 3 = 4$.

(c) Use the definition of $x^{\underline{m}}$ to simplify the following fraction: $\frac{x^{\underline{m+1}} - x^{\underline{m}}}{x^{\underline{m}} + x^{\underline{m+1}}}$

We have: $\frac{x^{\underline{m+1}} - x^{\underline{m}}}{x^{\underline{m}} + x^{\underline{m+1}}} = \frac{x(x-1)(x-2)\dots(x-m) - x(x-1)(x-2)\dots(x-m+1)}{x(x-1)(x-2)\dots(x-m+1) + x(x-1)(x-2)\dots(x-m)}$

$= \frac{x(x-1)(x-2)\dots(x-m+1)(x-m-1)}{x(x-1)(x-2)\dots(x-m+1)(1+x-m)} = \frac{x-m-1}{1+x-m} = \frac{x-m-1}{x-m+1}$

(d) Present the proof of Theorem 1 step-by-step by applying Equation 6.

Step 1 : Equation 6 states that: $\sum_a^b u(x)\, \Delta v(x)\, \delta_x = u(x)v(x) \Big|_a^{b+1} - \sum_a^b Ev(x)\, \Delta u(x)\, \delta_x$

Step 2 : let $a = 0$, $b = n-1$, $u(k) = \mathbb{F}_k$ and $\Delta v(k) = 1$

Then we have $\Delta u(k) = \mathbb{F}_{k+1} - \mathbb{F}_k = \sum_{i=1}^{k+1} \frac{1}{F_i} - \sum_{i=1}^{k} \frac{1}{F_i} = \frac{1}{F_{k+1}}$, $v(k) = k$

and $Ev(k) = v(k+1) = k+1$

Step 3 : then we obtain: $\sum_{k=0}^{n-1} \mathbb{F}_k = \mathbb{F}_k \cdot k \Big|_0^n - \sum_{k=0}^{n-1} (k+1) \cdot \frac{1}{F_{k+1}} = n\mathbb{F}_n - \sum_{k=0}^{n-1} \frac{k+1}{F_{k+1}}$

We complete our proof

(e) Show that $\mathbb{F}_n^{(r)} - \mathbb{F}_{n-2}^{(r)} = \mathbb{F}_n^{(r-1)} + \mathbb{F}_{n-1}^{(r-1)}$.

Step 1 : From the definition of $\mathbb{F}_n^{(r)}$ we have: $\mathbb{F}_n^{(r)} = \sum_{k=1}^{n} \mathbb{F}_k^{(r-1)}$

Step 2 : Then $\mathbb{F}_n^{(r)} - \mathbb{F}_{n-2}^{(r)} = \sum_{k=1}^{n} \mathbb{F}_k^{(r-1)} - \sum_{k=1}^{n-2} \mathbb{F}_k^{(r-1)} = \mathbb{F}_n^{(r-1)} + \mathbb{F}_{n-1}^{(r-1)}$

Therefore, we have the desired result.

(f) Determine the Euclidean norm of the circulant matrix Circ$(1, 1, 0, 0)$.

Step 1: Let $C = \mathrm{Circ}(1,1,0,0)$ then C contains 4 rows, each rows has two 1's and two 0's. We can also write $C = (c_{ij})$

Step 2: Then $\|C\|_E = \left(\sum_{i=1}^{4}\sum_{j=1}^{4} |c_{ij}|^2\right)^{\frac{1}{2}} = (8.|1|^2 + 8.|0|^2)^{\frac{1}{2}} = 8^{\frac{1}{2}} = 2\sqrt{2}$.

(g*) Show that for $u(k) = \mathbb{F}_k^2$ we get $\Delta u(k) = \frac{1}{F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)$.

Step 1 : From the definition of $\mathbb{F}_k$ we have $\mathbb{F}_{k+1} = \sum_{i=1}^{k+1} \frac{1}{F_i} = \mathbb{F}_k + \frac{1}{F_{k+1}}$

Step 2 : We have $\Delta u(k) = u(k+1) - u(k) = \mathbb{F}_{k+1}^2 - \mathbb{F}_k^2 = (\mathbb{F}_{k+1} - \mathbb{F}_k)(\mathbb{F}_{k+1} + \mathbb{F}_k)$

$= \left(\mathbb{F}_k + \frac{1}{F_{k+1}} - \mathbb{F}_k\right)\left(\mathbb{F}_k + \frac{1}{F_{k+1}} + \mathbb{F}_k\right) = \frac{1}{F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)$

(h*) Use the theorems from the article to prove the following identity:

$$\sum_{k=1}^{n-1} k^m (\mathbb{F}_k)^2 = \frac{n^{m+1}}{m+1}\mathbb{F}_n^2 - \sum_{k=0}^{n-1} \frac{(k+1)^{m+1}}{(m+1)F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)$$

Step 1 : Applying equation 6 with $u(k) = \mathbb{F}_k^2$ and $\Delta v(k) = k^m$

Step 2 : Then from theorem 2, theorem 4 and (g*) we have:

$\Delta u(k) = \frac{1}{F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)$, $v(k) = \frac{k^{m+1}}{m+1}$, $Ev(k) = \frac{(k+1)^{m+1}}{m+1}$. We obtain the desired result.

(i*) Use Equation 1 and Theorem 5 to show the following:

$$\sum_{k=0}^{n-1} \frac{\mathbb{F}_k}{k+1} = \mathbb{F}_n + \sum_{k=0}^{n-1}\left(\frac{\mathbb{F}_n H_k}{n} - \frac{H_{k+1}}{F_{k+1}}\right)$$

Step 1 : Theorem 5 states that: $\sum_{k=0}^{n-1} \frac{\mathbb{F}_k}{k+1} = H_n \mathbb{F}_n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}}$

Step 2 : Equation 1 states that: $\sum_{k=1}^{n-1} H_k = nH_n - n \Rightarrow H_n = 1 + \sum_{k=1}^{n-1} \frac{H_k}{n}$

Step 3 : Then we have:

$\sum_{k=0}^{n-1} \frac{\mathbb{F}_k}{k+1} = \mathbb{F}_n\left(1 + \sum_{k=1}^{n-1} \frac{H_k}{n}\right) - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}} = \mathbb{F}_n + \sum_{k=0}^{n-1}\left(\frac{\mathbb{F}_n H_k}{n} - \frac{H_{k+1}}{F_{k+1}}\right)$ (since $H_0 = 0$)